On rolling, tunneling and decaying in some large N vector models

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ABSTRACT: Various aspects of time-dependent processes are studied within the large N approximation of O(N) vector models in three dimensions. These include the rolling of fields, the tunneling and decay of vacua. We present an exact solution for the quantum conformal case and find a solution for more general potentials when the total change of the value of the field is small. Characteristic times are found to be shorter when the time dependence of the field is taken into account in constructing the exact large N effective potentials. We show that the different approximations yield the same answers in the regions of the overlap of the validity. A numerical solution of this potential reveals a tunneling in which the bubble that separates the true vacuum from the false one is thick.

Contents

1 Introduction

Evolutionary processes were studied in particle physics during the whole time of its own evolution. However, in recent years the study of time-dependent processes has played an increasing role in cosmology as well as Quantum Field Theory. The spot light has shifted away from the study of vacua and perturbations around it. Metastable vacua and their properties are investigated in detail. In many cases an understanding of the time-dependent processes is important. In addition to the interesting theoretical issues involved, the problem emerged in cosmology in general, and in the cosmology which deals with a notion of a landscape of vacua in particular. The time-dependent processes determine both the tunneling to a different potential well and the rolling of a field down to the region of a minimum of a potential. Similar physical processes appear also in some field theory models of possible phenomenological interest, like for instance in the supersymmetric models with

long-living metastable vacua in which SUSY is broken [1] . Also in the Standard Model and beyond it such a tunneling followed by a rolling towards the minimum can occur.

In most cases a detailed study of these processes is difficult. Systems accounting for gravity and ignoring it were studied. Various approximations have been used, such as the semiclassical and the thin wall approximations [2–4] .

Understanding the exact features of such time-dependent processes is important for theory and may eventually have practical uses.

For this purpose we conduct a study in the case of large N vector models in three space-time dimensions.

Traditionally, quantum field theories with N dynamical variables, $N \to \infty$, have served mostly as study grounds for extending one's intuition in handling basic problems of quantum field theories [5]. Approximate results obtained at large N possess many of the properties believed to be true in an exact solution. Such processes have been studied in the large N approximation of the vector model in $d = 4$. Here we obtain new results and highlight additional aspects for $d = 3$, which includes a quantum conformal case.

In this paper we investigate various time-dependent processes in this model. In section 2 we review the rich phase structure of the three dimensional vector models and the methods of solving them. We next calculate in section 3 an exact large N effective action which allows treating time-dependent solutions. In the derivation we take into account time variations also in the potential terms. We derive the general equations which govern an evolution of the system and solve them in two special cases. In the first case (section 4) we consider a rolling of the field from the top of a potential ϕ^6. The corresponding theory is also quantum conformal [6], and it makes a solution accessible. We show that the field continues to escape to infinity in a finite time. A time is shorter, however, than that obtained in an approximation where the time dependent effects in the potential are not accounted for. This situation is of interest in studying possible holographic descriptions of Big Crunch singularities [7] .

In the second case (section 5) we consider a rolling in a general potential in an approximation of small changes in fields along the evolution. We derive a characteristic time of run away from a potential maximum and a frequency of small oscillations around a minimum. The frequency of oscillations is bigger than the corresponding frequency obtained for an effective potential derived assuming static solutions only.

We also study a possibility of a tunneling in the system (section 6). We show that a solution in this case looks like an expanding bubble of a complicated shape different than that resulting in the thin-wall approximation [2] in the semiclassical case. We illustrate this fact by a numerical example.

We end with a set of appendices.

2 Scalar model in the large N limit — A brief review

Let us consider an $O(N)$ symmetric Euclidean action for an N - component scalar field $\vec{\phi}$ in three space-time dimensions

$$S\left(\vec{\phi}\right) = \int \left[\frac{1}{2}\left(\partial_\mu \vec{\phi}\right)^2 + NU\left(\frac{\vec{\phi}^2}{N}\right)\right] d^3x\,. \tag{2.1}$$

The potential has a Taylor expansion of the form

$$U\left(\frac{\vec{\phi}^2}{N}\right) = \sum_{n=1}^{\infty} \frac{g_{2n}}{2n}\left(\frac{\vec{\phi}^2}{N}\right)^n , \tag{2.2}$$

with g_{2n} kept fixed as $N \to \infty$ (in the large N limit any such potential is renormalizable [8]). We describe a way to find the generating functional of this model. It is given by

$$Z\left[\vec{J}\right] = \int D\vec{\phi}\exp\left[-S(\vec{\phi}) - \int \vec{J}(x)\cdot\vec{\phi}(x)d^3x\right] . \tag{2.3}$$

Inserting

$$1 \sim \int D\rho\delta(\vec{\phi}^2 - N\rho) \sim \int D\rho D\lambda e^{-i\int \frac{\lambda}{2}(\vec{\phi}^2-N\rho)d^3x} \tag{2.4}$$

and integrating over $\vec{\phi}$ one obtains

$$Z\left[\vec{J}\right] = \int D\rho D\lambda\, e^{-NS_{\text{eff}}(\rho,\lambda)}e^{\frac{1}{2}\int \vec{J}(x)(-\Box+i\lambda)_{xy}^{-1}\vec{J}(y)d^3x d^3y} , \tag{2.5}$$

where

$$S_{\text{eff}}(\rho,\lambda) = \frac{1}{2}\int \left[2U(\rho) - i\lambda\rho\right]d^3x + \frac{1}{2}Tr\ln\left(-\Box + i\lambda\right) . \tag{2.6}$$

When N is large, the last form suggests using the saddle point method to calculate the integral. The two saddle point equations, obtained by varying the auxiliary fields ρ and λ are[1]

$$2U'(\rho) = i\lambda, \qquad \rho = tr\frac{1}{-\Box + i\lambda} = -\frac{\sqrt{i\lambda}}{4\pi} \equiv -\frac{m}{4\pi} , \tag{2.7}$$

where m will assume a role of a mass. Moreover, we have used the dimensional regularization procedure in order to define the divergent loop. One can also define an effective potential whose minimum fixes a value of the mass

$$\frac{U_{\text{eff}}}{N} = \frac{m^3}{24\pi} + \sum_{n=1}^{\infty}\frac{g_{2n}}{2n}\left(-\frac{m}{4\pi}\right)^n . \tag{2.8}$$

In the above analysis and in what follows we take into account the fact that there is no spontaneous breaking of the $O(N)$ symmetry. The complete vacuum energy [5, 6] which accounts for $O(N)$ broken phase is given by

$$\frac{U_{\text{eff}}}{N} = \frac{m^3}{24\pi} + \sum_{n=1}^{\infty}\frac{g_{2n}}{2n}\left(\frac{\vec{\phi}_c^2}{N} - \frac{m}{4\pi}\right)^n , \tag{2.9}$$

where $\vec{\phi}_c$ is the would be expectation value of $\vec{\phi}$.

When one considers only relevant and marginal terms up to ϕ^6, then the potential is stabilized for $0 \le g_6 \le (4\pi)^2$. It is unstable for $g_6 < 0$, the instability implied for $g_6 > g_c$ is discussed in the next section. If only g_6 is present the theory is conformal and possesses two $O(N)$ invariant phases, one with $g_6 < (4\pi)^2$ and $m = 0$, with m being the mass, and another with $g_6 = (4\pi)^2$ and arbitrary m.

In the following sections we generalize the saddle point equations (2.7) to the case when the fields ρ and λ are not constant.

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[1]We use here the definition $Tr = \int d^3x\, tr$

3 Case of non-constant fields — General approach

In this section we introduce a set of equations that govern evolutionary processes in the theory with Euclidean action given in (2.1) in the large N limit. These equations are

$$2U'\big(\rho(x)\big) = i\lambda(x), \qquad \rho(x) = G_{\mathrm{reg}}(x,x), \qquad \Big(-\Box_x + i\lambda(x)\Big) G(x,y) = \delta(x-y). \quad (3.1)$$

Here $G(x,y)$ is the Green's function and $G_{\mathrm{reg}}(x,x)$ is the regularized Green's function

$$G_{\mathrm{reg}}(x,x) = \lim_{x \to y}\left(G(x,y) - \frac{1}{4\pi|x-y|}\right). \qquad (3.2)$$

In order to derive these equations recall that, as described in section 2, any constant solution of the theory (2.1) in the large N limit is a solution of the gap equations (2.7)

$$2U'(\rho) = i\lambda, \qquad \rho = tr\frac{1}{-\Box + i\lambda} \qquad (3.3)$$

The trace in the second equation is $tr\frac{1}{-\Box+i\lambda} = G(x,x)$, where $G(x,y)$ is a propagator of a scalar field in a background $i\lambda$ which solves the last equation in (3.1). The regularization required for the divergent $G(x,x)$ leads to (3.2). The above equations remain valid also for non-constant fields. Any solution of these equations will describe some process in the theory. Constant solutions described in section 2 are examples of solutions, but there are many others. Some examples will be given in what follows.

The main difficulty in finding solutions stems from the fact that the last equation in (3.1) involves computing a Green's function of the operator with an unknown function $\lambda(x)$. Since, to the best of our knowledge, there is no closed expression for such a Green's function, we shall consider each case separately.

Yet another approach is suggested in [9, 10], where the $O(N)$ $g_4\phi^4$ theory in four dimensions is explored in a large-N limit. In particular, the effective action and the corresponding gap equations are derived within the subspace of slowly varying functions $\lambda(x)$ for which the inequality $|(\partial\lambda)^2/\lambda^3| \ll 1$ holds. In appendix A we build upon this idea to construct also an effective action in this approximation and analyze different scenarios of dynamical evolution within it.

4 Time-dependent solution in the conformal case

In the first case we consider the effective potential (2.2) which contains only g_6 - the dimensionless coefficient, the bare potential is thus

$$U = \frac{g_6}{6\,N^2}\vec{\phi}^{\,6}, \qquad (4.1)$$

whereas the full quantum effective potential according to (2.8) is

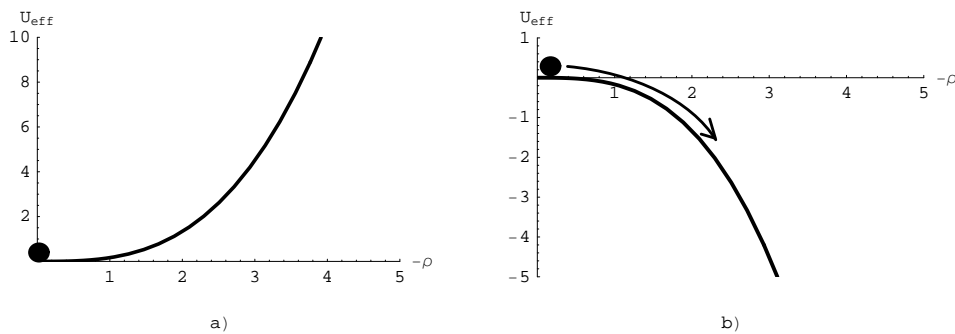$$U_{\mathrm{eff}}(\rho) = \frac{g_6 - g_c}{6}\rho^3, \qquad (4.2)$$

Figure 1. a) Effective potential for $g_6 = g_c - 1$, b) Effective potential for $g_6 = g_c + 1$.

where, as before, $\rho = \langle \vec{\phi}^2 \rangle / N$ and g_c is the critical value of g_6

$$g_c = (4\pi)^2. \tag{4.3}$$

As mentioned in section 2, the theory is scale invariant also quantum mechanically and possesses two different phases: one with unbroken scale invariance for $g_6 < g_c$, and the other with spontaneously broken scale invariance and a spontaneously generated mass for $g_6 = g_c$. The behavior of the system depends on the sign of g_6 in the classical case and in addition on whether g_6 is bigger or smaller than g_c in the quantum case. Note that from (2.7) it follows that relevant values of ρ after a regularization are negative. The quantum potential for both signs of $g_6 - g_c$ and some types of motion in it that will be considered below are shown in figure 1. This potential plays a role in various attempts to obtain a holographic dual to a gravitational system which may be suspect of exhibiting a Big Crunch [7] .

The unstable potential resulting for $g_6 > g_c$ can be misleading. In that case the UV cutoff Λ can not be removed, a UV completion is needed to have a cutoff independent theory. In this paper we check what would have been the time scales involved if one insisted to keep the unstable potential as is, with neither a cutoff nor a UV completion. This is done to learn about the properties of such unstable potentials when they actually arise.

In this section we present a solution to equations (3.1) in the scale invariant phase. We consider a situation when all fields depend only on one coordinate out of three, and we choose it to be a time t. The scale invariance of the potential dictates the asymptotic form of the time-dependent solution. Initial conditions such as the initial time t_0 and the initial value of the field at time t_0 break scale invariance and are thus reflected in the solution as well. The solution will require two integration constants.

The solution that we find is actually the classical motion described by the full quantum effective action $\Gamma(\vec{\phi})$. We compare this solution with its classical counterpart, by which we mean a classical motion in the quantum effective potential (4.2) with all corrections that include derivatives of the field neglected. Since $\rho \sim \phi^2$ this last case looks as a purely classical problem of the motion in the potential ϕ^6. In what follows we will call this motion a classical motion, in contrast with the motion described by the full action $\Gamma(\vec{\phi})$, which we will call the quantum one.

Here we study a solution for which the classical energy vanishes. This initial condition does not break scale invariance. However, a second initial condition, for example, the value of the field ϕ_0 at the time t_0, does introduce a scale. We compute the time evolution of the field for both the classical and quantum cases for zero energy and show that although qualitatively the two cases are similar there is a quantitative difference: the quantum rolling is faster. If one considers a time which takes the field to get from a given initial value to infinity in a way that the total energy vanishes then the precise relation between these times in the classical and quantum cases is

$$\Delta t_q = \frac{\Delta t_{\rm cl}}{\sqrt{3}} \tag{4.4}$$

A solution for the non vanishing energy in the classical case is presented in appendix B.

4.1 The classical case

We consider here a classical motion of the field in the effective potential (4.2). We start with the following Lagrangian

$$L = \frac{1}{2} \left(\partial_\mu \vec{\phi} \right)^2 - \frac{a}{6N^2} \left(\vec{\phi}^2 \right)^3 , \tag{4.5}$$

where a is an arbitrary constant. For time-dependent and space-independent solutions, the Lagrangian is

$$L = \frac{1}{2} \dot{\vec{\phi}}^2 - \frac{a}{6N^2} \left(\vec{\phi}^2 \right)^3 , \tag{4.6}$$

where dot means a derivative w.r.t. t. The point $\vec{\phi}^2 = 0$ in the space of fields corresponds to the extremum of the potential. In what follows we mostly consider an inverted conformal potential and investigate those classical solutions with vanishing total energy, which either asymptotically tend to $\vec{\phi}^2 = 0$ or increase indefinitely. Such solutions are closely related to the exact quantum ones which we are going to find in the next subsection, in a sense that for both cases (classical and quantum) the total energy is zero, the system approaches the equilibrium point $\vec{\phi}^2 = 0$ as $t \to \pm\infty$ and the field's magnitude diverges after a finite time.

The general ansatz for any classical solution of the considered problem is given by

$$\vec{\phi} = \sqrt{N} \phi(x) \hat{n} , \tag{4.7}$$

where $\phi(x)$ is an arbitrary function and $\hat{n}$ is a unit vector in the space of fields. For the class of solutions which we are interested in, $\phi(x)$ can be determined by the scaling arguments (see below). Moreover, one can show that $\hat{n} = const$ within the subspace of solutions under consideration. Substituting the above ansatz into the Lagrangian and bearing in mind that $\hat{n} = const$, one obtains

$$L = N \left(\frac{1}{2} \dot{\phi}^2 - \frac{a}{6} \phi^6 \right) . \tag{4.8}$$

The corresponding EOM is

$$\ddot{\phi} = -a\phi^5 . \tag{4.9}$$

By scaling arguments the solution of vanishing energy should be of the form

$$\phi = \frac{\gamma}{\sqrt{\pm(t - t_{\rm div})}} \ , \qquad \rho_{\rm cl} = \pm \frac{\gamma^2}{t - t_{\rm div}} \ , \tag{4.10}$$

where $t_{\rm div}$ is a time when the field may diverge and $\rho_{\rm cl}$ is an analog of ρ, which is $\vec{\phi}^2/N$. For a positive value of the coefficient a of the ϕ^6 potential the only zero energy solution is that in which the field is stuck statically at the minimum of the potential. To obtain truly time-dependent solutions with zero energy one needs to study the case of the potential unbounded from below obtained for $a < 0$. In this case the EOM determines the value of γ to be

$$\gamma = \sqrt[4]{-\frac{3}{4a}} \ . \tag{4.11}$$

Also in this case, there are two different kinds of time-dependent solutions with zero energy: the field can either escape towards infinity or asymptotically tend to the maximum of the potential. We will consider the first case, which implies that in the solution (4.10) there should be $t < t_{\rm div}$ and the sign should be minus. Insisting on placing the field at the maximum of the potential at time t_0 will result with a static solution remaining there. In order to obtain a time-dependent solution we can set either ϕ_0 or $\dot\phi_0$ to be non-zero. The one determines the other through

$$\gamma = \sqrt{\frac{\phi_0^3}{2\dot\phi_0}} \ . \tag{4.12}$$

With these initial conditions the solution is

$$\phi = \frac{\gamma}{\sqrt{t_0 - t + \frac{\gamma^2}{\phi_0^2}}} \ , \qquad \rho_{\rm cl} = \frac{\gamma^2}{t_0 - t + \frac{\gamma^2}{\phi_0^2}} \ , \tag{4.13}$$

with γ as in (4.11). From here one reads off that

$$t_{\rm div} = t_0 + \frac{\gamma^2}{\phi_0^2} \ . \tag{4.14}$$

That means that the field starting at a finite value ϕ_0 at time t_0 reaches an infinite value after a finite time $\Delta t_{\rm cl}$ which is

$$\Delta t_{\rm cl} = \frac{1}{2\phi_0^2} \sqrt{-\frac{3}{a}} \ . \tag{4.15}$$

In particular, for the potential (4.2) this time is

$$\Delta t_{\rm cl} = \frac{1}{2\rho_0} \sqrt{\frac{3}{g_6 - g_c}} \ . \tag{4.16}$$

This case can be interpreted as a calculating the divergence time in an exact quantum effective action in which, however, no account was taken yet of the time derivatives of the fields beyond the standard kinetic term. We next turn to consider the effect of this time dependence.

4.2 The full quantum solution

Here we consider the full quantum evolution with all time derivative corrections included. As in the previous section, we consider a process with vanishing energy. Similarly to the classical case, the scale invariance fixes the time dependence of the functions

$$\rho(t) = \frac{\bar{\alpha}_L}{t - t_{\text{div}}} , \qquad i\lambda(t) = \frac{\alpha_L}{(t - t_{\text{div}})^2} . \qquad (4.17)$$

where the subscript L stands for "Lorentzian." In this section we compute the coefficients $\bar{\alpha}_L$ and α_L. We show that in addition to the trivial solution $\bar{\alpha}_L = \alpha_L = 0$ there exists in the case $g_6 > g_c$ another one

$$\bar{\alpha}_L = \frac{1}{2\sqrt{g_6 - g_c}} , \qquad \alpha_L = \frac{1}{4}\frac{g_6}{g_6 - g_c} . \qquad (4.18)$$

If $g_6 > g_c$ then the quantum potential is unbounded from below in the sense described before, and the nontrivial solution represents a rolling from the maximum of the potential towards infinity. However, if $0 < g_6 < g_c$ then the potential has a minimum at $\rho = 0$ with vanishing energy. Since we consider only solutions with zero energy, we find only a trivial one, corresponding to a field at the bottom of the potential well.

From this one can compute, as for the classical case, a rolling time Δt_q, the time it takes the field to roll from a given value of the field ρ_0 to an infinite value, according to the solution (4.17) with coefficients (4.18); this time turns out to be

$$\Delta t_q = \frac{1}{2\,\rho_0\sqrt{g_6 - g_c}} . \qquad (4.19)$$

One can compare this rolling time with a similar time Δt_{cl} for a rolling in the effective potential (4.2) with ρ replaced by ϕ^2, as explained in the previous section. The result is

$$\Delta t_q = \frac{\Delta t_{\text{cl}}}{\sqrt{3}} , \qquad (4.20)$$

the quantum rolling is faster than the corresponding classical one. It should be stressed that there is no reason for the two times to be equal, since the effective potential captures only those terms in the full quantum effective action which do not involve time derivatives of the field beyond the canonical kinetic term; those terms do however appear in the case of non-constant solutions. We will see another example of this phenomenon in the next section, where we consider a rolling in a general renormalizable potential.

We start all computations in the Euclidean signature. The scale invariant ansatz for the functions ρ and $i\lambda$ is

$$\rho(\tau) = \frac{\bar{\alpha}_E}{\tau - \tau_{\text{div}}} , \qquad i\lambda(\tau) = \frac{\alpha_E}{(\tau - \tau_{\text{div}})^2} , \qquad (4.21)$$

where τ is a Euclidean time and τ_{div} is a Euclidean time instance at which the field diverges. The subscript E emphasizes that the work in the Euclidean space-time. In what follows we will choose τ_{div} to be zero. This choice fixes a time translation symmetry of the problem.

The Lorentzian motion is recovered by replacing τ by it. This, in turn, leads to the expression (4.17) for ρ with $\bar{\alpha}_L = -i\,\bar{\alpha}_E$. The field ρ is real if $\bar{\alpha}_E$ is imaginary. Since, as explained in section 3, ρ is a value of the regularized Green's function with coincident points, the full Green's function must be complex.

The next step is to check if there is a non real-valued Green's function of the operator $-\Box + \alpha_E/\tau^2$. There are two different cases: 1) $\alpha_E < -1/4$, 2) $\alpha_E > -1/4$. The Green's functions for both cases are derived in appendix C.

In the case $\alpha_E > -1/4$ the Green's function is

$$G(\mathbf{r}, \tau, \tau_0) = \frac{\sqrt{|\tau|\,|\tau_0|}}{2\pi} \frac{1}{W_+ W_-} \left(\frac{W_+ - W_-}{W_+ + W_-}\right)^\beta , \tag{4.22}$$

where

$$\beta = \frac{\sqrt{1 + 4\alpha_E}}{2} , \qquad W_\pm = \sqrt{\mathbf{r}^2 + (|\tau| \pm |\tau_0|)^2} , \tag{4.23}$$

and the Euclidean times τ and τ_0 must be of the same sign. This Green's function is real and therefore does not lead to a real-valued field ρ. This case corresponds to the potential ϕ^6 with a positive coefficient, which is bounded from below. In this case the only solution with zero energy is the trivial one $\rho = 0$.

Turn now to the case $\alpha_E < -1/4$, which, as will be shown in course of the derivation, corresponds to the potential unbounded from below. In this case β in (4.23) becomes imaginary and the Green's function (4.22) becomes complex-valued.[2] The short distance expansion of this function (for $\mathbf{r} \to 0$ and $\tau \to \tau_0$) is

$$G(\mathbf{r}, \tau, \tau_0) \simeq \frac{1}{4\pi\sqrt{\mathbf{r}^2 + (|\tau| - |\tau_0|)^2}} - \frac{\beta}{4\pi|\tau_0|} . \tag{4.24}$$

The first term is a singularity of precisely the form anticipated in (3.2), and it is canceled in the regularized Green's function, which therefore is

$$G_{\mathrm{reg}}(\tau, \tau) = -\frac{\beta}{4\pi|\tau|} \equiv -i\frac{\sqrt{|1 + 4\alpha_E|}}{8\pi|\tau|} . \tag{4.25}$$

Since according to (3.1) $G_{\mathrm{reg}}(\tau, \tau) \equiv \rho(\tau)$ we get

$$\bar{\alpha}_E = -i\frac{\sqrt{|1 + 4\alpha_E|}}{8\pi} \mathrm{sign}\,\tau . \tag{4.26}$$

The value is imaginary, and therefore after the Wick rotation the solution is real. The sign of τ which appears in this expression reflects the fact that, as in the classical case, positive or negative τ's describe the field that either runs to infinity or tends asymptotically to zero. From the first equation in (3.1), which in this case is

$$g_6\,\rho^2(\tau) = i\lambda(\tau) , \tag{4.27}$$

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[2]As explained in appendix C, in this case there is also another real-valued Green's function. It describes a $\rho = 0$ solution in the potential unbounded from below.

we conclude that

$$\alpha_E = \frac{1}{4} \frac{g_6}{g_c - g_6} \ . \tag{4.28}$$

We see that $\alpha_E < -1/4$ holds for $g_6 > g_c$, and this is precisely the case when the effective potential is unbounded from below and one expects to have a rolling solution. The coefficient $\bar{\alpha}_E$ defined in (4.26) is

$$\bar{\alpha}_E = -\frac{i}{2\sqrt{g_6 - g_c}} \operatorname{sign} \tau \ . \tag{4.29}$$

Wick rotate now to the Lorentzian signature, $\tau = i\,t$. The Lorentzian solution is the analytic continuation of our result

$$G_{\mathrm{reg}}(t,t) \equiv \rho(t) = -\frac{\beta}{4\pi\, i|t|} \equiv -\frac{\sqrt{|1 + 4\alpha_E|}}{8\pi |t|} \ . \tag{4.30}$$

We see that ρ is negative, as necessary. The evolution of the field is either a rolling from the top of the effective potential, which has no minimum in this case, for $t < 0$, or a running towards zero for $t > 0$. We will concentrate on the case of rolling. Plugging in the value of α_E from (4.28) and taking $t < 0$ we obtain

$$\rho(t) = \frac{1}{2\,t\,\sqrt{g_6 - g_c}}, \qquad t < 0 \ . \tag{4.31}$$

This, after restoring t_{div}, leads to the following final form of the rolling solution

$$\rho(t) = \frac{\bar{\alpha}_L}{t - t_{\mathrm{div}}} \ , \qquad i\lambda(t) = \frac{\alpha_L}{(t - t_{\mathrm{div}})^2} \ , \tag{4.32}$$

where

$$\bar{\alpha}_L = \frac{1}{2\sqrt{g_6 - g_c}} \ , \qquad \alpha_L = \frac{1}{4} \frac{g_6}{g_6 - g_c} \ , \tag{4.33}$$

and we have restored the divergence time t_{div}.

If g_6 is sufficiently close to g_c, then all the fields involved in the problem are slowly varying functions of time and the approximation suggested in [9] is applicable. We build upon this approximation in appendix A in order to validate the exact results obtained here. We find a full agreement.

Moreover, we want to compare these results with the classical ones of the previous section. In order to do it we consider the quantum process which is analogous to the process considered there: the field starts at certain value ρ_0 at the initial time t_0 and reaches the infinity after some time Δt_q. This rolling time is

$$\Delta t_q = \frac{1}{2\rho_0} \frac{1}{\sqrt{g_6 - g_c}} \ . \tag{4.34}$$

Comparing this rolling time with its classical counterpart (4.16) we see that the quantum rolling is faster

$$\Delta t_q = \frac{\Delta t_{\mathrm{cl}}}{\sqrt{3}} \ . \tag{4.35}$$

As explained above, the two times need not be equal because in calculating Δt_q one takes into account terms in the effective action that contain derivatives of the field (in addition to the usual kinetic term). These terms are not taken into account in the classical rolling.

In appendix D we show that one can get a feeling whether quantum effects tend to accelerate or decelerate a rolling. The leading order in $\hbar$ correction to the force is $-\sigma^2 U'''(\phi)/2$, where σ is the standard deviation. In particular, in our case according to equation (4.8) the correction is $-10\,a\,\sigma^2\phi^3$. We see that the classical force $-a\,\phi^5$ and the quantum correction are of the same sign, and therefore the quantum correction increases the force. So indeed the quantum rolling should be faster. In the computation we have determined the precise relation.

4.3 Energy conservation

For completeness we illustrate the energy conservation for the process under consideration. The energy-momentum tensor in the Euclidean signature is given here by

$$T_{\mu\nu}(x) = \partial_\mu \vec{\phi}\, \partial_\nu \vec{\phi} - \delta_{\mu\nu} \left[\frac{1}{2}(\partial_\mu \vec{\phi})^2 + \frac{g_6}{6N^2}(\vec{\phi}^2)^3 \right]. \tag{4.36}$$

The expectation value of the energy-momentum tensor is calculated by noting that to leading order in the $1/N$ expansion

$$< (\vec{\phi}^2)^3 > = N^3 \rho^3\,, \tag{4.37}$$

and that

$$< \partial_\mu \vec{\phi}(x)\partial_\nu \vec{\phi}(y) > = N \frac{\partial^2}{\partial x^\mu \partial y^\nu} G(x,y)\,. \tag{4.38}$$

Using (4.22) we get the following regularized expressions

$$\lim_{\mathbf{r},\delta\tau\to 0} \frac{\partial^2}{\partial r^2} G(\mathbf{r},\tau,\tau_0) = \lim_{\mathbf{r},\delta\tau\to 0} \frac{1}{r}\frac{\partial}{\partial r} G(\mathbf{r},\tau,\tau_0) = \frac{\beta(1-\beta^2)}{12\pi\tau^3}, \tag{4.39}$$

$$\lim_{\mathbf{r},\delta\tau\to 0} \frac{\partial^2}{\partial\tau\partial\tau_0} G(\mathbf{r},\tau,\tau_0) = \frac{\beta(4\beta^2-7)}{48\pi\tau^3}, \tag{4.40}$$

where $\delta\tau \equiv \tau - \tau_0$.

In what follows we need to compute only the $T_{00}(x)$ since we are interested to demonstrate only the energy conservation (recall that we consider field configurations which depend only on time). First note that due to rotational and translational invariance in the $\mathbf{r}$-plane one has

$$< \partial_i \vec{\phi}(x)\partial^i \vec{\phi}(y) > = N \frac{\partial^2}{\partial x^i \partial y_i} G(\mathbf{r},\tau,\tau_0) = -N \frac{\partial^2}{\partial x^i \partial x_i} G(\mathbf{r},\tau,\tau_0) = -N \triangle G(\mathbf{r},\tau,\tau_0)$$

$$= -N \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} \right) G(\mathbf{r},\tau,\tau_0)\,. \tag{4.41}$$

As a result, combining (4.21), (4.36)–(4.41) altogether, we finally obtain

$$\frac{\langle T_{00} \rangle}{N} = \frac{1}{2}\lim_{\mathbf{r},\delta\tau\to 0} \left(\frac{\partial^2}{\partial\tau\partial\tau_0} G(\mathbf{r},\tau,\tau_0) + \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} \right) G(\mathbf{r},\tau,\tau_0) \right) - \frac{g_6}{6}\rho^3 = 0\,. \tag{4.42}$$

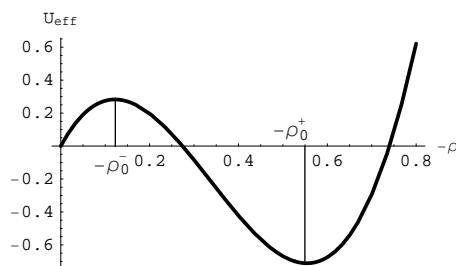The zero value of the energy of the system is indeed conserved during the process considered.

Figure 2. Effective potential (5.1) for $g_2 = -10$, $g_4 = -100$ and $g_6 = 10$. The field ρ is negative after the renormalization

5 Case of small changes of fields

We next consider a case of renormalizable potential with non-vanishing coefficients g_2, g_4 and g_6 (the restriction to this particular renormalizable potential is not necessary and can be omitted without any loss of generality). For our needs we choose $g_2 < 0$, $g_4 < 0$, $0 < g_6 < g_c$, where g_c is defined in (4.3). The corresponding effective potential according to (2.8) is

$$U_{\text{eff}} = \frac{g_2}{2}\rho + \frac{g_4}{4}\rho^2 + \frac{g_6 - g_c}{6}\rho^3 \ . \tag{5.1}$$

In this expression the field ρ is constrained to be negative. The potential is drawn in figure 2. It has a metastable ground state at the origin and stable minimum away from the origin.

In this section we describe another approach to obtaining time-dependent solutions. It is valid in an approximation in which the field does not deviate too much from the extremum of the potential. The potential (5.1) possesses two extrema

$$\rho_0^\pm = \frac{g_4}{2(g_c - g_6)}\left(1 \pm \sqrt{1 + \frac{4g_2}{g_4^2}(g_c - g_6)}\right) \ . \tag{5.2}$$

These extrema are shown in figure 2.

We study the solution rolling from the top of the potential down to its minimum and the solution which oscillates around the minimum. We derive a characteristic time of runaway from the top of the potential ρ_0^- as well as the frequency of small oscillations around the true minimum ρ_0^+. These will turn out to be determined by the equations

$$\arctan \frac{\omega}{8\pi\rho_0^\mp} = \frac{4\pi}{g_4 + 2g_6\rho_0^\mp}\,\omega \ . \tag{5.3}$$

In the case of oscillations this equation determines the frequency ω, whereas for the rolling it gives the inverse rolling time $t \sim 1/\omega$. Relying on (5.2) one can show explicitly that this equation possesses real solutions along with imaginary ones, depending on the sign before the square root in (5.2). We explain their physical meaning below.

We also compare the results with the corresponding classical counterparts for the same effective potential in the approximation when the two extrema are close to each other

$$\left|\frac{\rho_0^+ - \rho_0^-}{\rho_0^+}\right| \ll 1 \ . \tag{5.4}$$

We find that within this approximation the frequency of the quantum oscillations ω_q is larger than that of the classical ones ω_{cl}

$$\omega_q \simeq \sqrt{3}\,\omega_{\mathrm{cl}} \ . \tag{5.5}$$

The characteristic quantum rolling time for t_q is smaller than its classical counterpart t_{cl}

$$t_q \simeq \frac{t_{\mathrm{cl}}}{\sqrt{3}} \ . \tag{5.6}$$

We turn now to the derivation of these results. For the choice of potential equations (3.1) are given by

$$g_2 + g_4\rho(x) + g_6\rho^2(x) = i\lambda(x), \quad \rho(x) = G_{\mathrm{reg}}(x,x), \quad \Big(-\Box_x + i\lambda(x)\Big)G(x,y) = \delta(x-y) \ . \tag{5.7}$$

Let us assume that the total change of ρ and λ in the course of dynamical evolution is small compared to their mean values. This is the case if one would explore the system during short time intervals. It also holds during the whole evolution process if one imposes condition (5.4).

We expand the fields as

$$\begin{aligned}
\delta\rho(x) &= \rho(x) - \rho_0 = C\rho_1(x) + C^2\rho_2(x) + \dots, \\
\delta\lambda(x) &= \lambda(x) - \lambda_0 = C\,\lambda_1(x) + C^2\,\lambda_2(x) + \dots,
\end{aligned} \tag{5.8}$$

where ρ_0 and $i\lambda_0$ provide a constant solution to (5.7) and C is an expansion parameter which is assumed to be small: $|C| \ll 1$. As mentioned in section 2, in the case of a constant solution at the minimum of the potential, the value of $i\lambda$ coincides with a square of a physical mass, and therefore

$$i\lambda_0 = m^2 \ . \tag{5.9}$$

The last equation in (5.7) can be written as

$$\Big(-\Box_x + m^2 + i\delta\lambda(x)\Big)G(x,y) = \delta(x-y) \ . \tag{5.10}$$

The solution to this equation can be found by performing a series expansion

$$G(x,y) = G_m(x,y) - \int d^3z\, G_m(x,z)\, i\delta\lambda(z)\, G_m(z,y) +$$
$$+ \int d^3z d^3w\, G_m(x,z)\, i\delta\lambda(z)\, G_m(z,w)\, i\delta\lambda(w)\, G_m(w,y) + \dots, \tag{5.11}$$

where $G_m(x,y)$ is a propagator of a free massive scalar field

$$G_m(x,y) = \frac{e^{-m|x-y|}}{4\pi|x-y|} \ . \tag{5.12}$$

One needs to regularize Green's function at coincident points. In the series expansion (5.11) only the first term diverges in the limit $x \to y$, and its regularized value is

$$G_{m,reg}(x,x) = \lim_{x\to y}\left(\frac{e^{-m|x-y|}}{4\pi|x-y|} - \frac{1}{4\pi|x-y|}\right) = -\frac{m}{4\pi} \ . \tag{5.13}$$

This result, in particular, shows that $\rho_0 = -m/4\pi$ in accordance with the known result presented in section 2. Altogether, the expression for ρ becomes

$$\rho(x) = -\frac{m}{4\pi} - \int d^3z \, G_m^2(x, z) \, i\delta\lambda(z) +$$
$$+ \int d^3z \, d^3w \, G_m(x, z) \, i\delta\lambda(z) \, G_m(z, w) \, i\delta\lambda(w) \, G_m(w, x) + \dots \quad (5.14)$$

One can solve equations (5.7) order by order in C.

- *Order 0.* To this order equations (5.7) yield

$$g_2 + g_4 \rho_0 + g_6 \rho_0^2 = m^2, \qquad \rho_0 = -\frac{m}{4\pi}, \qquad (5.15)$$

where, as above, $i\lambda_0 = m^2$. There are two solutions:

$$\rho_0^{\pm} = \frac{g_4}{2(g_c - g_6)} \left(1 \pm \sqrt{1 + \frac{4g_2}{g_4^2}(g_c - g_6)} \right), \qquad i\lambda_0^{\pm} = g_c \, \rho_0^{\pm} \ . \qquad (5.16)$$

- *Order 1.* In this order one has to compute the integral in the second term of the r.h.s. in (5.14). It involves a convolution of G_m^2 and $i\lambda_1$, and the result is

$$\rho_1(x) = \frac{1}{4\pi\sqrt{-\Box}} \arctan \frac{\sqrt{-\Box}}{8\pi\rho_0} \, i\lambda_1(x) \ . \qquad (5.17)$$

The first equation in (5.7) then gives

$$\left(g_4 + 2\rho_0 g_6 \right) \rho_1(x) = \frac{4\pi\sqrt{-\Box}}{\arctan \frac{\sqrt{-\Box}}{8\pi\rho_0}} \, \rho_1(x) \ . \qquad (5.18)$$

If we look for a solution which depends on a single coordinate, a Euclidean time τ, this equation simplifies

$$\left(g_4 + 2\rho_0 g_6 \right) \rho_1(\tau) = \frac{4\pi \, i\partial_\tau}{\arctan \frac{i\partial_\tau}{8\pi\rho_0}} \, \rho_1(\tau), \qquad (5.19)$$

and its form suggests the following solution

$$\rho_1(\tau) = e^{i\omega\tau} \, , \qquad (5.20)$$

where ω satisfies equation (5.3).

The result for ω depends on the choice of ρ_0: ω is real for $\rho_0 = \rho_0^-$ and imaginary for $\rho_0 = \rho_0^+$. This means the following: ρ_0^- corresponds to a maximum of the potential and the result describes a runaway solution, whereas ρ_0^+ is a minimum of the potential and in that case ω describes the frequency of the oscillations around it (recall that the computation was carried out in a Euclidean time and in order to get a Lorentzian solution one has to Wick rotate the time, which introduces an additional factor i in the exponents).

The solution for $i\lambda$ to this order is

$$i\lambda_1(\tau) = A_1 \, e^{i\omega\tau}, \qquad A_1 = \frac{4\pi\omega}{\arctan \frac{\omega}{8\pi\rho_0}} \equiv g_4 + 2g_6 \, \rho_0 \ . \qquad (5.21)$$

One can continue the perturbative expansion to higher orders. There will appear corrections both to the frequency of oscillations and to the shape of the solution. This follows from the fact that if the amplitude is large enough the field will overshoot the value ρ_0^- and there will be no oscillations around it.

Let us look more closely at the frequency ω of small oscillations around ρ_0^+. It is determined by equation (5.3). Assume that the two extrema of the potential are close to each other, as in equation (5.4). This approximation will be valid if the following condition holds

$$\frac{4g_2}{g_4^2}(g_c - g_6) \gtrsim -1 . \tag{5.22}$$

Due to the assumption about the coupling constants ($g_2 < 0$, $g_4 < 0$, $0 < g_6 < g_c$), this criterion can be satisfied. In this case ω is small and thus it can be determined by a simpler equation obtained by expanding the arctangent to the third order

$$\omega_q^2 \simeq -\frac{3\,g_4^2}{g_c - g_6} \sqrt{1 + \frac{4g_2}{g_4^2}(g_c - g_6)} . \tag{5.23}$$

This is the Euclidean result, the Lorentzian one will be of opposite sign. The Euclidean result for ω_q^2 is negative for $g_6 < g_c$, and therefore the Lorentzian result will be positive, as expected. In appendix A we rederive this result in the framework of another approximation presented in [9]. We find full agreement between the results.

This frequency can be compared to a frequency $\omega_{\rm cl}$ of oscillations of a classical field ϕ in an effective potential (5.1). Since the potential is $O(N)$ invariant the most rapid oscillations occur in the radial direction, thus the maximal value of the classical frequency is determined by the equation $\omega_{\rm cl}^2 = U''(\phi_0)$, where $\phi_0 = \sqrt{\rho_0^+}$ is the value of the field ϕ at the extremum. In our approximation the Euclidean result is

$$\omega_{\rm cl}^2 \simeq -\frac{g_4^2}{g_c - g_6} \sqrt{1 + \frac{4g_2}{g_4^2}(g_c - g_6)} . \tag{5.24}$$

Therefore

$$\omega_q \simeq \sqrt{3}\,\omega_{\rm cl} , \tag{5.25}$$

and we conclude that the quantum field oscillates faster around the true minimum than the classical one.

One can carry out a similar computation concerning the characteristic time of escape from the potential maximum at ρ_0^-. The result is similar: the escape time in the quantum case is shorter than its classical counterpart by the same factor of $\sqrt{3}$.

6 Tunneling

In this section we investigate a possible tunneling in the system. Following Coleman [2] we search for solutions which depend on a Euclidean radius

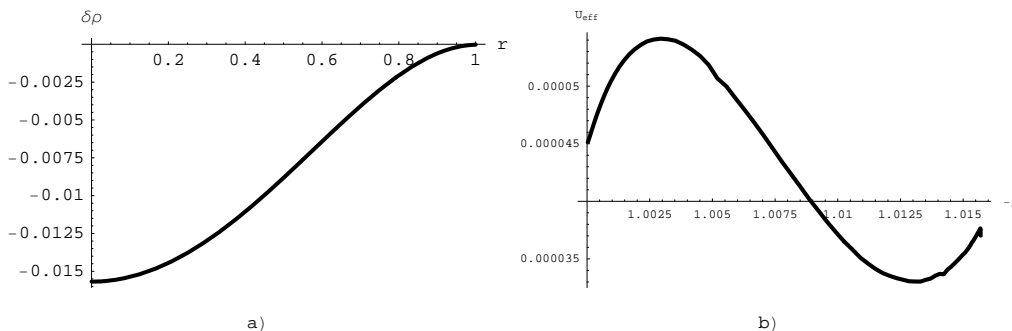$$r_E = \sqrt{\vec{r}^2 + \tau^2}, \tag{6.1}$$

Figure 3. a) Function $\delta\rho(r)$ giving a deviation from the false vacuum which is at $r = \infty$, b) Resulting effective potential U_{eff} shifted by a constant -26.3189 $[m]^3$.

where $\vec{r}$ is a distance in space and τ is a Euclidean time. The physical meaning of these solutions is revealed after the Wick rotation back to the Lorentzian time. The rotated solution depends on a combination $r_L = \sqrt{\vec{r}^2 - t^2}$, where t is a Lorentzian time. Such a solution describes a spherical wave in the space time. Thus, for example, within the thin wall approximation of the field evolution after the tunneling, the real and positive values of r_L correspond to points where the wall has not yet arrived, whereas imaginary values correspond to those points where it has already passed.

In what follows we will work in the Euclidean signature and will suppress the subscript E.

Relying on the aforementioned feature [8] that three dimensional theory in the large N limit is renormalizable regardless the shape of the potential, we base our approach on the reverse engineering method: we will derive the equation for the instanton, then we will choose a solution which describes a tunneling in some unknown effective potential, and at the end we will reconstruct numerically the potential itself. Our consideration is reliable if the false and the true vacua are close to each other.

Our main results are summarized in figure 3, where figure 3a is a plot of a function which describes a deviation of the solution from its value at the false vacuum as a function of the Lorentzian radius $r = \sqrt{\vec{r}^2 - t^2}$ for its real values ($r = \infty$ corresponds to the false vacuum itself), and figure 3b is a plot of the effective potential with false and true vacua.

We also compute the tunneling amplitude and compare it to the corresponding semi-classical value. In both cases the amplitude is $A \simeq \exp(-N S)$. In the semiclassical case[3] $S_{\text{scl}} \simeq 4.7$, whereas in the full quantum computation $S \simeq 3.56$, so the quantum amplitude turns out to be larger.

We consider a situation when the change in fields during the tunneling process is small

$$\left|\frac{\rho_t - \rho_f}{\rho_t}\right| \ll 1, \tag{6.2}$$

where ρ_t and ρ_f are the values of ρ in the true and false vacuum respectively. We start with an arbitrary potential $U(\rho)$ and carry out the solution which presents a combined

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[3]The thin wall approximation is not assumed in the numerical analysis of the semiclassical case.

numerical description of a tunneling and a subsequent rolling. The numerical analysis is done relying on the equations which are valid up to a leading order in the aforementioned small parameter.

Write similarly to (5.8)

$$\rho(x) \simeq -\frac{m}{4\pi} + C\rho_1(x), \qquad i\lambda(x) = m^2 + C\,i\lambda_1(x) \ . \tag{6.3}$$

To order 0 we will have similarly to (5.15)

$$2U'(\rho_0) = m^2, \qquad \rho_0 = -\frac{m}{4\pi} \ . \tag{6.4}$$

The order 1 equation (5.17) also remains intact. The two equations can be combined into a single one

$$2U'\big(\rho(x)\big) = -(4\pi)^2 \rho_0^2 + \frac{4\pi\sqrt{-\Box}}{\arctan\frac{\sqrt{-\Box}}{8\pi\rho_0}}\,\rho(x) \ . \tag{6.5}$$

This is a general form of the equation which describes a behavior of the system to the leading non-trivial order. In the case that we want to consider, namely, when the field ρ depends only on the Euclidean radius r the equation simplifies. As we show in appendix E, on functions depending only on the radius, the operator $\sqrt{-\Box}$ becomes

$$\sqrt{-\Box}\,\rho(r) = -\hat{A}\,\rho(r) = -\frac{1}{\pi r}PV\int\limits_{-\infty}^{\infty}\frac{\big(s\,\rho(s)\big)'}{s-r}\,ds, \tag{6.6}$$

where PV denotes the Cauchy principal value and we assume that the function $\rho(r)$ has been continued to the region of negative r as an even function

$$\rho(-r) = \rho(r) \ . \tag{6.7}$$

Equation (6.6) also serves as a definition of an operator $\hat{A}$ which will be extensively used in what follows.

From what was said above we arrive at the final form of the equation for $\rho(r)$

$$2U'\big(\rho(r)\big) = -(4\pi)^2 \rho_0^2 + \frac{4\pi\,\hat{A}}{\arctan\frac{\hat{A}}{8\pi\rho_0}}\,\rho(r) \ . \tag{6.8}$$

This equation should be supplemented by boundary conditions for $\rho(r)$. Following [2] we require that $\rho(r)$ go its value at the false vacuum when $r \to \infty$. We denote this value of ρ by ρ_0 and define

$$\delta\rho(r) = \rho(r) - \rho_0 \ . \tag{6.9}$$

The function $\delta\rho(r)$ goes to 0 as r goes to infinity. With this definition equation (6.8) becomes

$$2U'\big(\rho_0 + \delta\rho(r)\big) - 2U'(\rho_0) = \frac{4\pi\hat{A}}{\arctan\frac{\hat{A}}{8\pi\rho_0}}\,\delta\rho(r), \qquad 2U'(\rho_0) = (4\pi)^2 \rho_0^2 \ . \tag{6.10}$$

Linearized around ρ_0

$$U''(\rho_0)\,\delta\rho(r) = \frac{2\pi\hat{A}}{\arctan\frac{\hat{A}}{8\pi\rho_0}}\,\delta\rho(r)\ . \tag{6.11}$$

Note, that since at ρ_0 the potential has a minimum $U''(\rho_0)$ is positive. However, as shown in appendix E, all eigenvalues of $\hat{A}$ are negative, and, since $\rho_0 < 0$, it follows that the operator on r.h.s. of (6.11) is negatively definite. Therefore there are no solutions to equation (6.11). Nevertheless, this feature may not pose a problem since one cannot extrapolate it to the solution of equation (6.10).

One can rewrite equation (6.10) in the following form

$$U''(\rho_0)\,\delta\rho(r) + \sigma\big(\delta\rho(r)\big) = \hat{B}\,\delta\rho(r), \qquad \hat{B} = \frac{4\pi\hat{A}}{\arctan\frac{\hat{A}}{8\pi\rho_0}}, \tag{6.12}$$

where the function $\sigma\big(\delta\rho(r)\big)$ is proportional to $\delta\rho^2(r)$. Since for large real r the function $\delta\rho(r)$ goes to 0 we conclude that it should become an eigenfunction of $\hat{B}$ at least for very large r.

We say that a function $f(r)$ is an "asymptotic eigenfunction" of the operator $\hat{B}$ if there is a number k (an "asymptotic eigenvalue") such that f obeys the following requirement

$$\lim_{r\to\infty} f(r) = 0, \qquad \lim_{r\to\infty} \frac{\hat{B}\,f(r) - k\,f(r)}{f(r)} = 0\ . \tag{6.13}$$

In appendix E we show that the operator $\hat{B}$ possesses many asymptotic eigenfunctions with positive asymptotic eigenvalues (see (E.15) and a discussion there and remember that $\rho_0 < 0$). Consider the following illustrative example.

The construction of the candidate function $\delta\rho$ starts with a function $f_1(r)$

$$f_1(r) = \begin{cases} 5r(r^2-1)^2, & r < 1 \\ 0, & r > 1 \end{cases} \tag{6.14}$$

This function is plotted in figure 4a. Without any loss of generality, let us take both the eigenvalue and ρ_0 to be equal -1. Then one can compute (see appendix E for details) the functions $\delta\rho$ and $\hat{B}\delta\rho$

$$\delta\rho(r) = -\frac{2}{\pi}\int\limits_0^\infty \Psi(k)\frac{\arctan\frac{k}{8\pi}}{4\pi k + \arctan\frac{k}{8\pi}}\frac{\sin kr}{r}\,dk\,,$$

$$\hat{B}\delta\rho(r) = \frac{2}{\pi}\int\limits_0^\infty \Psi(k)\frac{4\pi k}{4\pi k + \arctan\frac{k}{8\pi}}\frac{\sin kr}{r}\,dk\,, \tag{6.15}$$

where $\Psi(k)$ is the sine-Fourier transform of $f_1(r)$

$$\Psi(k) = \int_0^\infty f_1(r)\sin(kr)dr = \frac{40k(k^2-15)\cos k - 120(2k^2-5)\sin k}{k^6}\ . \tag{6.16}$$
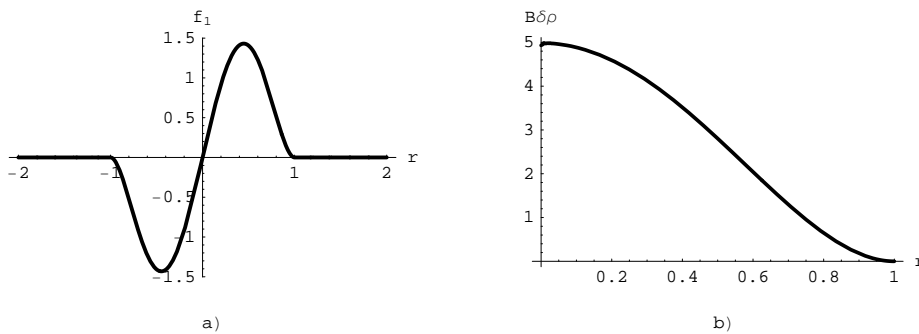
– 18 –

Figure 4. a) Function $f_1(r)$, as defined in (6.14), b) Function $\hat{B}\delta\rho(r)$.

These functions are plotted in figure 3a and 4b. One can also reconstruct the classical potential from equation (6.10), and then, using the definition of the effective potential (2.8)

$$U_{\rm eff}(\rho) = U(\rho) - \frac{(4\pi)^2}{6}\rho^3 \,, \tag{6.17}$$

one can reconstruct it as well. The result is presented in figure 3b. We see that the potential indeed increases from its value at $\rho = 1$ (the fact that its derivative does not vanish at this point is a numerical error, the numerics is not reliable at this region; the derivative is very small though). We see also that the effective potential has a minimum at $\rho \simeq 1.012$ and then grows up. Since $\rho - \rho_0 \ll \rho$ the approximation is valid along the process .

These results have the following interpretation. At $t = 0$ the field tunnels from the false vacuum and a bubble is created. Its shape is given by the function $\rho(r)$ that we computed, and there are points in space where the field acquires all values that are covered by the plot of the potential 3b. Then, as the time grows, the solution evolves according to the law $\rho(\sqrt{\vec{r}^2 - t^2})$. We see that the initial shape of the bubble is complicated, contrary to the case of the thin wall approximation.

The fact that our numerical computation reconstructs the minimum of the potential and even goes further means the following. In the semiclassical approximation the computation similar to the one presented here would reconstruct the potential up to a point which is just a little beyond the point where the value of the potential is equal to its value at the false vacuum. That the field at all gets beyond the turning point (a point where the potential is precisely equal to its value at the false vacuum) is because of the friction term in the EOM. If there is no friction term, like in the case of quantum mechanics, the computation will only be able to reconstruct the potential up to its turning point. Our computation gets so far beyond the turning point and even beyond the true vacuum because the action we work with is very complicated. In particular, it possesses a non-standard kinetic term, which, in turn, leads to a non-trivial friction.[4]

Having computed the potential, one can also evaluate the effective action on the solution in order to get the tunneling amplitude. In the case under consideration the action is $S_{\rm eff} \simeq 3.56$, and the tunneling amplitude is $\exp(-N\,S_{\rm eff})$.

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[4] We are grateful to J. Barbon for discussions on these issues.

This result can be compared with a prediction of the semiclassical approximation for the same potential. In that computation one has to solve a radial Euclidean EOM for a bounce and then to evaluate its action. The numerical result in this case is $S_{\rm sc} \simeq 4.7$, and the tunneling amplitude, similarly to the case above, is $\exp(-N\, S_{\rm eff})$. We see that the full quantum tunneling amplitude is larger in this case. Once again this falls within the pattern that taking account of time-dependent effects shortens the characteristic time of the process.

7 Conclusions

We have studied a variety of time-dependent processes. These included rolling of fields, tunneling among vacua and their decay. This was done for a large class of vector $O(N)$ models in three space time dimensions, among them the system which is conformal also quantum mechanically. Using the methods of the large N expansion we were able to take into account the effects of the time variation of the fields on the exact effective potentials. We have calculated exact and approximate characteristic time scales of such processes. In most cases the results were quantitatively different than those obtained without considering the time dependency in the effective potential. A qualitative difference is found when analyzing the bubble driving the decay of a false vacuum. The bubble shows thick rather than thin characteristics. For the cases studied a pattern of accelerated time scales emerged. The next natural step is to study supersymmetric extensions of such systems and their coupling to gravity.

Acknowledgments

A Effective action in the region of slowly varying fields

In this appendix we follow [9] in order to compute the effective action which governs the dynamics of the large-N theory within the subspace of slowly varying functions $\lambda(x)$, for which the inequality $|(\partial\lambda)^2/\lambda^3| \ll 1$ holds. We compare the results obtained for the exact conformal case and those obtained for small field variations. The comparison is done in those regions where the validity of the approximations overlap.

For simplicity of notation we suppress imaginary unit i in front of λ in what follows, that is $i\lambda \to \lambda$.

Let us define $\Gamma(\lambda)$ from the Euclidean functional integral

$$e^{-\Gamma(\lambda)} = \int D\vec{\phi} \exp\left(-\int d^3x \left[\frac{1}{2}\vec{\phi}(-\Box + \lambda)\vec{\phi}\right]\right)$$
$$= \exp\left[-\frac{N}{2}Tr\ln(-\Box + \lambda)\right] . \tag{A.1}$$

In the regime of aforementioned approximation $\Gamma(\lambda)$ can be written as a local expansion

$$\Gamma(\lambda) = \int d^3x \left[F_0(\lambda) + F_1(\lambda)(\partial_\mu\lambda)^2 + \ldots\right] , \tag{A.2}$$

where $F_0(\lambda)$ and $F_1(\lambda)$ are local functions of $\lambda(x)$. In particular, $F_0(\lambda)$ is found by calculating the $Tr\ln(-\Box + \lambda)$ with a constant λ. Up to an infinite constant, which is zero in the dimensional regularization, we have

$$\frac{N}{2}Tr\ln(-\Box + \lambda) = \frac{N}{2}\int \frac{d^3x\,d^3p}{(2\pi)^3}\ln\left(1 + \frac{\lambda}{p^2}\right)$$
$$= -\frac{N}{12\pi}\int d^3x\,\lambda^{3/2} , \tag{A.3}$$

and thus

$$F_0(\lambda) = -\frac{N}{12\pi}\lambda^{3/2} . \tag{A.4}$$

On the other hand, in order to calculate $F_1(\lambda)$ one notes that

$$-\frac{\delta^2\Gamma(\lambda)}{\delta\lambda(x)\delta\lambda(y)} = \frac{1}{4}\left[\langle\vec{\phi}^2(x)\vec{\phi}^2(y)\rangle - \langle\vec{\phi}^2(x)\rangle\langle\vec{\phi}^2(y)\rangle\right]$$
$$= \frac{N}{2}\int \frac{d^3k}{(2\pi)^3}e^{ik(x-y)}\int \frac{d^3p}{(2\pi)^3}\frac{1}{((p-k)^2 + \lambda)(p^2 + \lambda)}$$
$$= \frac{N}{16\pi}\int \frac{d^3k}{(2\pi)^3}\,e^{ik(x-y)}\int_0^1 d\alpha\left(\alpha(1-\alpha)k^2 + \lambda\right)^{-1/2} . \tag{A.5}$$

This can be expanded in the form

$$-\frac{\delta^2\Gamma(\lambda)}{\delta\lambda(x)\delta\lambda(y)} = \frac{N}{16\pi}\int \frac{d^3k}{(2\pi)^3}\,e^{ik(x-y)}\left(\lambda^{-1/2} - \frac{k^2}{12\pi}\lambda^{-3/2} + O(k^4)\right)$$
$$= \frac{N}{16\pi}\lambda^{-1/2}\delta(x-y) + \frac{N}{192\pi}\lambda^{-3/2}\Box\delta(x-y) + O(\Box^2\delta(x-y)), \tag{A.6}$$

where in the second equality we assume that λ is constant. Such an expansion is justified due to the assumed approximation regime of slowly varying fields. Using equation (A.2) we can now identify $F_0(\lambda)$ and $F_1(\lambda)$ with the constant-λ expansion of equation (A.6)

$$\frac{\partial^2 F_0(\lambda)}{\partial^2\lambda} = -\frac{N}{16\pi}\lambda^{-1/2},$$
$$F_1(\lambda) = \frac{N}{384\pi}\lambda^{-3/2} . \tag{A.7}$$

Integrating the first of these equations reproduces the result of (A.4). Thus, effective action (2.6) can be rewritten as follows

$$S_{\text{eff}}(\rho, \lambda) = \int d^3x \left[U(\rho) - \frac{\rho\lambda}{2} - \frac{\lambda^{3/2}}{12\pi} + \frac{\lambda^{-3/2}}{384\pi}(\partial\lambda)^2 \right] . \tag{A.8}$$

If we now define

$$\psi(x) = \frac{\lambda(x)^{1/4}}{\sqrt{12\pi}} , \tag{A.9}$$

then effective action can be written as follows

$$S_{\text{eff}}(\rho, \psi) = \int d^3x \left[U(\rho) - \frac{(12\pi)^2}{2}\rho\psi^4 - (12\pi)^2\psi^6 + \frac{1}{2}(\partial\psi)^2 \right] . \tag{A.10}$$

Let us exploit this action in order to illustrate the process of tunneling and rolling in the case when the potential $U(\rho)$ is given by

$$U(\rho) = \frac{g_2}{2}\rho + \frac{g_4}{4}\rho^2 . \tag{A.11}$$

Under this assumption the effective action $S_{\text{eff}}(\rho, \psi)$ turns out to be quadratic in auxiliary field ρ and thus integrating it out yields

$$S_{\text{eff}}(\psi) = \int d^3x \left[\frac{1}{2}(\partial\psi)^2 - \frac{(12\pi)^4}{4g_4}\psi^8 - (12\pi)^2\psi^6 + \frac{g_2}{2g_4}(12\pi)^2\psi^4 \right] . \tag{A.12}$$

Let us explore the temporal rolling first, that is we consider the situation when all the fields are time-dependent only. As a result, the corresponding Lorentzian equation of motion represents a particle of unit mass moving in a potential

$$V(\psi) = -\frac{(12\pi)^4}{4g_4}\psi^8 - (12\pi)^2\psi^6 + \frac{g_2}{2g_4}(12\pi)^2\psi^4 . \tag{A.13}$$

Thus,

$$E = \frac{1}{2}\left(\frac{d\psi}{dt}\right)^2 + V(\psi) \tag{A.14}$$

is a constant of the motion. This can be used to determine the qualitative features of the solutions by inspection.

As a simple example of rolling, consider the potential shown in figure 5 which corresponds to

$$-\left(\frac{g_4}{8\pi}\right)^2 < g_2 < 0, \ g_4 < 0 . \tag{A.15}$$

We are interested to investigate the rolling of the system from the local maximum of the potential situated at

$$\psi_-^2 = -\frac{g_4}{96\pi^2}\left(1 - \sqrt{1 + \left(\frac{8\pi}{g_4}\right)^2 g_2}\right) , \tag{A.16}$$
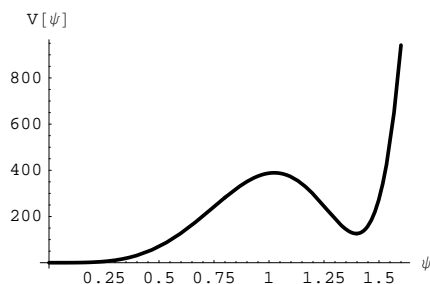
Figure 5. The effective potential (A.13) for $g_4 = -(12\pi)^2$, $g_2 = 0.91(g_4/8\pi)^2$.

down to the false vacuum located at

$$\psi_+^2 = -\frac{g_4}{96\pi^2}\left(1 + \sqrt{1 + \left(\frac{8\pi}{g_4}\right)^2 g_2}\right).\tag{A.17}$$

The corresponding runaway and oscillation frequencies are given respectively by

$$\omega_\pm^2 = \frac{d^2V}{d\psi_\pm^2}.\tag{A.18}$$

In this case, we are dealing with a solution of the equation of motion with $E = V(\psi_-)$, whence ψ as a function of t is given implicitly by

$$t = t_+ + \int_{\psi_+}^{\psi}\frac{d\psi}{\sqrt{2(V(\psi_-) - V(\psi))}},\tag{A.19}$$

where t_+ is an integration constant, the time at which ψ equals ψ_+.

On the other hand, according to Coleman [2], if we are interested to compute a decay probability per unit time per unit volume Γ/V, of the unstable state ψ_+ due to the barrier penetration, one must find the bounce $\bar{\psi}$, a solution of the Euclidean equations of motion

$$\frac{d^2\bar{\psi}}{dr^2} + \frac{2}{r}\frac{d\bar{\psi}}{dr} = V'(\psi)\tag{A.20}$$

subject to the following boundary conditions

$$\lim_{r\to\infty}\bar{\psi}(r) = \psi_+ \ , \ \bar{\psi}'(r = 0) = 0\,,\tag{A.21}$$

where prime denotes derivative with respect to Euclidean radius r. To leading order in $1/N$,

$$\Gamma/V \ \sim \ e^{-N S_{\text{eff}}(\bar{\psi})}(1 + O(1/N))\,.\tag{A.22}$$

In other words, in the limit of large N barrier penetration is exponentially small and thus, as emphasized in the text, the dynamics of the system is governed by the rolling processes only.

It is instructive to perform the calculation in the region where the approximation of slowly varying fields presented above overlaps with the small field approximation used in

the section 5. Therefore let us assume that condition (5.22) holds. As a result, the two extrema of the potential plotted on figure 2 can be made arbitrary close to each other, and the field derivatives in turn, during the evolution of the system from the top of the potential down to its minimum, become arbitrary small.

As an illustrative example, let us derive the frequency of small oscillations around the true minimum ρ_0^+. Relying on (A.10) the corresponding EOM are given by

$$\Box\psi = -2(12\pi)^2\rho\psi^3 - 6(12\pi)^2\psi^5\,,$$
$$(12\pi)^2\psi^4 = g_2 + g_4\rho + g_6\rho^2\,. \tag{A.23}$$

Since the oscillations are small, we linearize around the true vacuum

$$\Box\delta\psi = -2(12\pi)^2(3\rho_0^+\psi_0^{+\,2}\delta\psi + \psi_0^{+\,3}\delta\rho) - 30(12\pi)^2\psi_0^{+\,4}\delta\psi\,,$$
$$\delta\rho = \frac{4(12\pi)^2\psi_0^{+\,3}\delta\psi}{g_4 + 2g_6\rho_0^+}\,, \tag{A.24}$$

where

$$\psi_0^{+\,2} = -\frac{\rho_0^+}{3}\,,$$
$$\delta\rho = \rho - \rho_0^+\,,$$
$$\delta\psi = \psi - \psi_0^+\,. \tag{A.25}$$

As a result, the quantum frequency of oscillations is given by

$$\omega_q^2 = 12g_c\rho_0^{+\,2}\left[1 - \frac{2g_c\rho_0^+}{g_4 + 2g_6\rho_0^+}\right] \simeq -\frac{3\,g_4^2}{g_c - g_6}\sqrt{1 + \frac{4g_2}{g_4^2}(g_c - g_6)}\,. \tag{A.26}$$

This result agrees with (5.23) obtained via small field approximation.

The other case we will consider is that in which only g_6, the dimensionless coefficient, is present. Equations (A.23) in this case combine together and yield

$$\frac{\Box\psi}{\psi^5} = -6(12\pi)^2\left[1 - \sqrt{\frac{g_c}{g_6}}\right]\,. \tag{A.27}$$

Since in the region of slowly varying fields $1 \gg |(\partial\lambda)^2/\lambda^3| \sim |\Box\psi/\psi^5|$ one concludes that

$$\delta g = g_6 - g_c \ll 1 \tag{A.28}$$

in order to justify the approximation. This yields

$$\frac{\Box\psi}{\psi^5} = -27\,\delta g\,. \tag{A.29}$$

Therefore according to (4.10) the time-dependent and space-independent solution with vanishing energy is given by

$$\psi = \frac{\sqrt[4]{-3/(4a)}}{\sqrt{\pm(t - t_{\mathrm{div}})}} \tag{A.30}$$
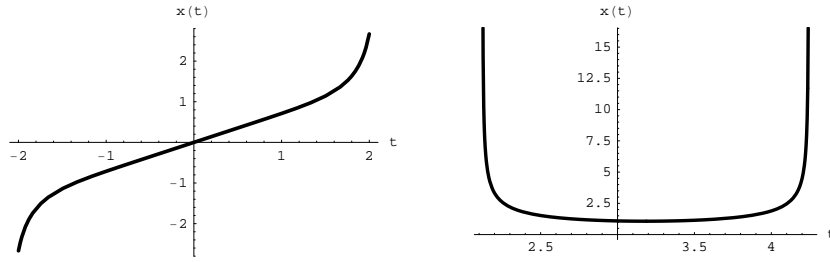
Figure 6. Solutions $x(t)$ for $a = 1$ and energies $E = \pm 0.25$.

with[5] $a = -27\,\delta g$.

Altogether

$$\lambda = (12\pi)^2\psi^4 \simeq \frac{g_6}{4(g_6 - g_c)}\frac{1}{(t - t_{\mathrm{div}})^2}\,, \tag{A.31}$$

which coincides with the exact results (4.17), (4.18).

B Classical motion in the ϕ^6 potential

In this appendix we present a solution which describes the classical evolution of a particle with energy E in the ϕ^6 potential. We derive the divergence time as a function of the energy of the particle.

Consider the following classical Lagrangian

$$L = \frac{1}{2}\dot{x}^2 + \frac{a}{6}x^6\,, \tag{B.1}$$

which describes a run-away potential (for positive a). A general solution is

$$x(t) = (9\sqrt{3} - 15)^{1/6}\left(\frac{E}{a}\right)^{1/6}\sqrt{\frac{1 - \mathrm{cn}(a^{1/6}E^{1/3}\xi, k)}{2 - \sqrt{3} + \mathrm{cn}(a^{1/6}E^{1/3}\xi, k)}}\,, \tag{B.2}$$

where

$$\xi = 2^{4/3}3^{1/12}(t - t_0), \qquad k = \frac{1}{4}(2 + \sqrt{3})\,, \tag{B.3}$$

and t_0 and E are integration constants. E is the total energy and t_0 is an arbitrary time. $\mathrm{cn}(x, k)$ is the elliptic cosine function. In figure 6 there are the solutions for $a = 1$ and $E = \pm 0.25$. We see that the solution with positive energy passes through 0, whereas the one with negative energy never reaches 0 (a particle comes from infinity and bounces back).

If a particle starts at some point x_0 at time t_0 with energy E then it gets to infinity after a time

$$\Delta t = \frac{1}{2^{4/3}3^{1/12}a^{1/6}E^{1/3}}\Bigg(\mathrm{cn}^{-1}(\sqrt{3} - 2, k) - $$
$$- \mathrm{cn}^{-1}\Big(\frac{3(3\sqrt{3} - 5)^{1/3}E^{1/3} + 2^{1/3}3^{1/6}(3 - 2\sqrt{3})\,a^{1/3}x_0^2}{3(3\sqrt{3} - 5)^{1/3}E^{1/3} + 2^{1/3}3^{2/3}\,a^{1/3}x_0^2}, k\Big)\Bigg)\,, \tag{B.4}$$

---

[5]Note that (4.10) corresponds to the solution of the Lorentzian EOM, whereas (A.23) corresponds to the Euclidean time. This reveals the origin of an extra minus sign in the expression for a.
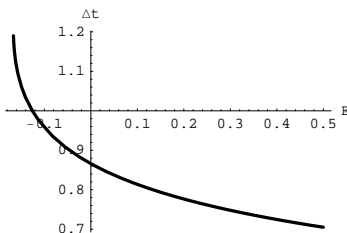
Figure 7. Divergence time as a function of energy for $a = 1$ and $x_0 = 1$.

where $\mathrm{cn}^{-1}(x, k)$ is the inverse elliptic cosine function. This divergence time is plotted as a function of energy in figure 7. We see that the more is the energy the less is the time that takes the particle to get to infinity. For small energies this divergence time is

$$\Delta t = \frac{1}{2x_0^2}\sqrt{\frac{3}{a}} - 0.67\frac{E}{x_0^8 a^{3/2}} \ , \tag{B.5}$$

and for vanishing energy one recovers (4.15).

C Green's function appearing in a conformal case

In this appendix we compute a regularized Green's function of the operator

$$L = -\Box + \alpha/\tau^2 \ .$$

This Green's function is used in section 4. For this we derive a short-distance expansion of the Green's function which solves the equation

$$L_x \, G(x, y) = \delta(x - y) \ . \tag{C.1}$$

The Green's function is given by the following expression:

$$G(x, y) = \sum_n \frac{\Psi_n(x)\Psi_n^*(y)}{\lambda_n}, \tag{C.2}$$

where Ψ's are eigenfunctions of L and λ's are corresponding eigenvectors.

A Laplacian in L is three-dimensional and involves a Euclidean time τ and two more coordinates which we will denote collectively by a vector $\mathbf{r}$. L possesses a translational invariance in the plane of $\mathbf{r}$ and therefore we are free to put the δ-function at any point in it, and we will choose this point to be the origin. It also occurs at some time τ_0. With this choice, the Green's function will depend only on the distance from the origin: $G(x, y) = G(\mathbf{r}, \tau, \tau_0)$, and only angular-independent eigenfunctions will contribute to the sum in (C.2). So, the relevant eigenfunctions of L are of the form $\Psi(x) = R(r)T(\tau)$. Here $R(r)$ satisfies the equation

$$-R''(r) - \frac{1}{r}R'(r) = \lambda_r R(r) \ . \tag{C.3}$$

Normalizable solutions of this equation exist only for positive λ_r and are given by

$$R(r) = \frac{1}{\sqrt{4\pi}} J_0(\sqrt{\lambda_r}\, r)\ . \tag{C.4}$$

The function $T(\tau)$ satisfies the equation

$$-T''(\tau) + \frac{\alpha}{\tau^2} T(\tau) = \lambda_\tau T(\tau)\ . \tag{C.5}$$

This equation was investigated in the context of Quantum Mechanics (with τ being a spacial coordinate) and is known to possess very different kinds of solutions for different values of α. We consider first the case of $\alpha > -1/4$. In this case λ_τ must be positive and the normalizable solutions are

$$T(\tau) = \sqrt{\frac{|\tau|}{2}}\, J_\beta(\sqrt{\lambda_\tau}\,|\tau|), \qquad \beta = \frac{\sqrt{1+4\alpha}}{2}. \tag{C.6}$$

In this case there are two disconnected Hilbert spaces for different signs of τ (since, in the language of Quantum Mechanics, there is no tunneling from a region of $\tau > 0$ to a region of $\tau < 0$ and back), and therefore the full propagator G will vanish unless τ and τ_0 are of the same sign, and this is what we assume from now on.

The full normalized eigenfunction of L is therefore

$$\Psi_{\lambda_r,\lambda_\tau}(r,\tau) = \sqrt{\frac{|\tau|}{8\pi}}\, J_\beta(\sqrt{\lambda_\tau}\,|\tau|)\, J_0(\sqrt{\lambda_r}\, r)\,, \tag{C.7}$$

and the corresponding eigenvalue is

$$\lambda = \lambda_r + \lambda_\tau\ . \tag{C.8}$$

According to equation (C.2), the Green's function is given by

$$G(\mathbf{r},\tau,\tau_0) = \frac{\sqrt{|\tau|\,|\tau|_0}}{8\pi} \int\limits_0^\infty \frac{d\lambda_r\, d\lambda_\tau}{\lambda_r + \lambda_\tau}\, J_0(\sqrt{\lambda_r}\, r)\, J_\beta(\sqrt{\lambda_\tau}\,|\tau|)\, J_\beta(\sqrt{\lambda_\tau}\,|\tau_0|)\ . \tag{C.9}$$

Using the integrals [12]

$$\int\limits_0^\infty \frac{dx}{x+y} J_0(r\sqrt{x}) = 2K_0(r\sqrt{y}), \tag{C.10}$$

$$\int\limits_0^\infty dx\, K_0(a\,x) J_\nu(b\,x) J_\nu(c\,x) = \frac{(r_+ - r_-)^\nu (r_+ + r_-)^{-\nu}}{r_+ r_-}, \qquad r_\pm = \sqrt{a^2 + (b \pm c)^2}, \tag{C.11}$$

we end up with the following Green's function of the operator L

$$G(\mathbf{r},\tau,\tau_0) = \frac{\sqrt{\tau\,\tau_0}}{2\pi} \frac{1}{W_+ W_-} \left(\frac{W_+ - W_-}{W_+ + W_-}\right)^\beta, \qquad W_\pm = \sqrt{\mathbf{r}^2 + (|\tau| \pm |\tau_0|)^2}\ . \tag{C.12}$$

This Green's function is real if β is real, which means, for $\alpha > -1/4$. If, however, $\alpha < -1/4$ then β is imaginary, and this Green's function is complex. This complex-valued Green's function is the one that describes a rolling in the ϕ^6 potential.

One can build a real Green's function in this case as well. In order to do this we use a fact that there is another solution to the Green's function equation (C.1), namely

$$\tilde{G}(\mathbf{r}, \tau, \tau_0) = \frac{\sqrt{|\tau|\,|\tau_0|}}{2\pi} \frac{1}{W_+ W_-} \left(\frac{W_+ + W_-}{W_+ - W_-} \right)^\beta . \tag{C.13}$$

This function is not appropriate to be a Green's function in the case of real and positive β since it diverges for small Euclidean times. However, for imaginary β this function is fine, and we can take a linear combination of it and (C.12) with equal coefficients to guarantee a real result. The sum of coefficients must be 1 for it to indeed be a solution to the Green's function equation. So, in the case of imaginary β we get the following real-valued Green's function

$$\tilde{G}(\mathbf{r}, \tau, \tau_0) = \frac{\sqrt{|\tau|\,|\tau_0|}}{2\pi} \frac{1}{W_+ W_-} \cos \left(|\beta| \log \left(\frac{W_+ + W_-}{W_+ - W_-} \right) \right) . \tag{C.14}$$

This Green's function gives a zero solution in an unbounded from below ϕ^6 potential.

D Rolling in quantum mechanics

In this appendix we build on the Ehrenfest theorem in order to compute the leading order quantum mechanical correction to the Newton's equations of motion. This in turn helps one to conclude whether quantum effects tend to accelerate or decelerate the classical rolling.

Ehrenfest theorem reveals a quantum mechanical generalization of Newton's second law. In particular, for a quantum particle moving in a one-dimensional potential $V(x)$ it states

$$m\frac{d^2}{dt^2}\langle \hat{x}(t) \rangle = -\langle \frac{d}{d\hat{x}} V(\hat{x}) \rangle , \tag{D.1}$$

where m is the mass of the particle and $\langle \hat{x}(t) \rangle$ is an expectation value of the position operator $\hat{x}(t)$.

Let us define

$$x(t) = \langle \hat{x}(t) \rangle ,$$
$$\hat{\eta}(t) = \hat{x}(t) - x(t) ,$$

then we get

$$\frac{dV}{d\hat{x}}(\hat{x}) = \frac{dV}{dx}(x(t) + \hat{\eta}) = \frac{dV}{dx} + \frac{d^2V}{dx^2}\hat{\eta} + \frac{1}{2}\frac{d^3V}{dx^3}\hat{\eta}^2 + \dots \tag{D.2}$$

Taking the expectation value of both sides yields

$$m\frac{d^2}{dt^2}x(t) = -\langle \frac{d}{d\hat{x}} V(\hat{x}) \rangle = -\frac{dV}{dx} - \frac{1}{2}\frac{d^3V}{dx^3}\sigma^2 + \dots , \tag{D.3}$$

where $\sigma = \sqrt{\langle \hat{\eta}^2 \rangle}$ is the standard deviation.

Thus, in order to figure out whether quantum mechanical corrections slow down or accelerate the rolling, one has to fix the relative sign between the expressions on the right hand side of the last identity. It turns out that this sign is not invariant and changes with the shape of potential.

E Operator $\sqrt{-\Box}$ on radial functions

In this appendix we derive a form of the operator $\sqrt{-\Box}$ when it acts on functions which depend only on a radius and discuss some related questions relevant to section 6.

If a function $\rho(x)$ depends only on the radius, then the operator $\sqrt{-\Box}$ simplifies. Indeed, consider the Fourier transform of $\rho(x)$. It depends only on the absolute value of the momentum p and therefore $\rho(x)$ in this case can be written as

$$\rho(x) = \int \frac{d^3p}{(2\pi)^3}\, \rho(p)\, e^{-ipx} = \frac{1}{2\pi^2 x} \int\limits_0^\infty dp\, p\, \rho(p) \sin p\, x\,, \tag{E.1}$$

where in the last equality we carried out the angular integration. The operator $\sqrt{-\Box}$ multiplies each Fourier mode by $|p|$, so it acts on $\rho(x)$ as

$$\sqrt{-\Box}\, \rho(x) = \frac{1}{2\pi^2 x} \int\limits_0^\infty dp\, p^2\, \rho(p) \sin p\, x\,. \tag{E.2}$$

Recall the definition of the Hilbert transform [11]

$$f(x) \to H[f](x) = \frac{1}{\pi} PV \int\limits_{-\infty}^\infty \frac{f(y)}{y - x} dy\,, \tag{E.3}$$

where PV stands for the Cauchy principal value of the integral. One of its main features is that it transforms $\sin p\, x$ to $\text{sign}(p) \cos p\, x$ and $\cos p\, x$ to $-\sin |p|\, x$. Using the Hilbert transform, the operator $\sqrt{-\Box}$ when it acts on a function that depends only on a radius can be written as

$$\sqrt{-\Box}\, \rho(x) = -\frac{1}{\pi x} PV \int\limits_{-\infty}^\infty \frac{(y\rho(y))'}{y - x}\, dy, \tag{E.4}$$

where we assumed that the function $\rho(x)$ is defined for negative x to be

$$\rho(-x) = \rho(x), \tag{E.5}$$

as is suggested also by equation (E.1).

Define, as in the body of the paper, an operator $\hat{A}$ to be

$$\hat{A}\, \rho(x) = \frac{1}{\pi x} PV \int\limits_{-\infty}^\infty \frac{(y\rho(y))'}{y - x}\, dy\,. \tag{E.6}$$

Find its eigenvalues and eigenfunctions. In order to do that recall the following features of the Hilbert transform: 1) its square is -1

$$H^2[f](x) \equiv H\big[H[f]\big](x) = -f(x), \tag{E.7}$$

and 2) it commutes with a derivative

$$H[f'](x) = H[f]'(x). \tag{E.8}$$

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Now consider the equation for eigenfunctions of our operator $\hat{A}$, which can be written as

$$\hat{A}f_k(x) \equiv \frac{1}{x}H\big[(xf_k)'\big](x) = kf_k(x), \tag{E.9}$$

where k is an eigenvalue and $f_k(x)$ is the corresponding eigenfunction. Acting on both sides with the same linear operator once again and using the properties of the Hilbert transform mentioned above, we arrive at the following differential equation

$$\frac{1}{x}\big(x\, f_k(x)\big)'' + k^2 f_k(x) = 0. \tag{E.10}$$

The solutions of this equation are $e^{\pm ikx}/x$. Since only functions which are finite at $x = \pm\infty$ are considered, k is real. By substituting these eigenfunctions into the equation (E.9) we find that the corresponding eigenvalue is negative and equals to $-|k|$. Thus one concludes that $\hat{A}$ is negative definite. The eigenfunctions that possess definite parity are

$$f_k^{(1)}(x) = \frac{\sin kx}{x}, \qquad f_k^{(2)}(x) = \frac{\cos kx}{x}. \tag{E.11}$$

Since we assume that $\rho(r)$ is even one has to drop off $f_k^{(2)}(x)$.

Next, we consider the operator $\hat{B}$ defined similarly to equation (6.12)

$$\hat{B} = \frac{\hat{A}}{\arctan\hat{A}}. \tag{E.12}$$

This operator is positive definite, as $\hat{A}$ is negative definite, and can be written as

$$\hat{B} = \frac{1}{x}\bar{B}\,x, \qquad \bar{B} = \frac{d}{\operatorname{arctanh} d}, \tag{E.13}$$

where d means the derivative w.r.t. x.

Our next goal is to both show that operator $\hat{B}$ possesses asymptotic eigenfunctions with negative eigenvalues, where the definition of such a function is given in (6.13), and to present a general procedure how to construct examples of them.

If $f(x)$ is an asymptotic eigenfunction of $\hat{B}$ with asymptotic eigenvalue k, then $g(x) := x\, f(x)$ obeys the following equation

$$\lim_{x\to\infty} \frac{\bar{B}\, g(x) - k\, g(x)}{g(x)} = 0. \tag{E.14}$$

Moreover, the requirement that $f(x)$ tends to zero at infinity can be replaced by the equivalent requirement that $g(x)$ defined above diverges at infinity slower than x.

Let us demonstrate how to construct the function $g(x)$ with the above properties and negative k. For simplicity of notation we take $k = -1$. By the definition of $g(x)$ one concludes that the function $f_1(x)$ defined by

$$f_1(x) = (\bar{B} + 1)g(x), \qquad k < 0 \tag{E.15}$$

might tend to zero at infinity faster than $g(x)$. From equation (E.5) it follows that both functions $g(x)$ and $f_1(x)$ have to be odd. Thus, taking an odd function $f_1(x)$ with compact support guarantees (E.14) and the following relation holds[6]

$$g(x) = (\bar{B} + 1)^{-1} f_1(x) = \frac{2}{\pi} \int\limits_0^\infty \Psi(k) \frac{\arctan k}{k + \arctan k} \sin kr \, dk \,, \qquad \text{(E.16)}$$

where $\Psi(k)$ is the sine-Fourier transform of $f_1(r)$

$$\Psi(k) = \int_0^\infty f_1(r) \sin(kr) dr \,. \qquad \text{(E.17)}$$

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[6]Note that the inverse operator $(\bar{B} + 1)^{-1}$ exists since as we have shown $\bar{B}$ is positive definite.

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